iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Bures fidelity for diagonalizable quadratic Hamiltonians in multi-mode systems

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2000 J. Phys. A: Math. Gen. 33 4925

(http://iopscience.iop.org/0305-4470/33/27/310)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.123
The article was downloaded on 02/06/2010 at 08:27

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. **33** (2000) 4925–4934. Printed in the UK PII: S0305-4470(00)12960-9

Bures fidelity for diagonalizable quadratic Hamiltonians in multi-mode systems

Xiang-Bin Wang, L C Kwek and C H Oh

Department of Physics, Faculty of Science, National University of Singapore, Lower Kent Ridge, Singapore 119260, Republic of Singapore

E-mail: scip7236@leonis.nus.edu.sg, scip6051@leonis.nus.edu.sg and phyohch@nus.edu.sg

Received 30 March 2000

Abstract. Fidelity, as a measure of the distinguishability of states, is an important concept in quantum mechanics, quantum optics and quantum information theory. Recently, the explicit expressions of fidelity for single-mode squeezed states have been given. However, in experimental studies, especially in non-degenerate parametric down-conversion, two photons are generated and one studies two- or more-mode systems. In this paper we study the Bures fidelity for thermal states of a diagonalizable quadratic Hamiltonian in multi-mode Fock space. To the best of our knowledge, no one has yet attempted to give an explicit general formula of fidelity of mixed states in multi-mode systems.

1. Introduction

A good quantum communication channel must be capable of transferring output quantum states which are close to the input states. To quantify this idea of closeness, it is often necessary to provide a measure to distinguish different quantum states. To do so, one introduces the idea of fidelity. A fidelity of unity implies identical states whereas a fidelity of zero implies orthogonal states. Indeed this idea of fidelity is not just confined to quantum communication. It is also important in quantum optics, quantum computing and quantum teleportation [1–4].

For pure states, this measure of fidelity is generally computed using the Hilbert–Schmidt norm. However, the Hilbert–Schmidt norm is not defined for mixed states. For mixed states, a good indicator of fidelity is the Bures fidelity [1], which is defined as

$$\left(\operatorname{tr}\sqrt{\hat{\rho}_1^{\frac{1}{2}}\hat{\rho}_2\hat{\rho}_1^{\frac{1}{2}}}\right)^2 \tag{1}$$

where $\hat{\rho}_1$ and $\hat{\rho}_2$ are density operators for two quantum mixed states. This fidelity not only satisfies the properties required for a measure of distinguishability of quantum states; it also reduces to the Hilbert–Schmidt fidelity for pure states.

Unlike the Hilbert–Schmidt norm, an explicit calculation of Bures fidelity is generally not easy because of a term involving taking the square root of an operator. Recently, by using a faithful representation, Twamley [5] calculated the Bures fidelity of a one-mode squeezed thermal state system. The result is significant because it is the first explicit expression of fidelity for mixed states in infinite-dimensional Fock space. Moreover, Twamley's results [6–8] were

0305-4470/00/274925+10$30.00 © 2000 IOP Publishing Ltd

subsequently re-derived using a group-theoretic approach and the matrix element of the square root operator was computed with an additional displacement term.

A single-mode squeezed state system can be produced in a degenerate parametric amplifier. However, in non-degenerate parametric down-conversion, one deals simultaneously with two photons in conjugate modes [9, 10]. The corresponding state can be generated from the vacuum by a two-mode squeezed operator

$$\hat{S}(\zeta) = e^{\zeta^* a_1^\dagger a_2^\dagger - \zeta a_1 a_2} \tag{2}$$

where a_i and $a_i^\dagger$ are annihilation and creation operators and ζ is a squeezed parameter (ζ^* being the complex conjugate of ζ). Indeed, experimentally two-mode squeezing can be achieved through a homodyne or heterodyne apparatus [10]. It is therefore natural to extend and investigate the multi-mode case and obtain a general formula for the fidelity of arbitrary multi-mode squeezed systems. In this article, we provide a systematic approach for the calculation of Bures fidelity for the multi-mode situation. In fact, the approach can be applied and modified to all systems involving a diagonalizable quadratic Hamilton in arbitrary modes. In section 2, we derive a general formula for a multi-mode system using a matrix representation method. In section 3, we apply our approach to compute the Bures fidelity of some non-trivial but instructive examples. In particular, we look at the one-dimensional generalized squeezed state system and the two-dimensional coupled harmonic oscillator system. Both systems are experimentally realizable. To our knowledge, nobody has so far provided an explicit formula for computing Bures fidelity for multi-mode systems. Moreover, the method involved in our generalization is also highly non-trivial. Before proceeding further, we define the following notations in an n-dimensional Fock space: $\alpha^{\mathrm{T}} = (a_1^\dagger, a_2^\dagger, \ldots, a_n^\dagger; a_1, a_2, \ldots, a_n)$ where $a_i^\dagger$ and a_i satisfy the commutation rules $[a_i, a_j^\dagger] = \delta_{ij}$ and α^T is the transpose of α.

2. General formula for Bures multi-dimensional Fock space

Quadratic Hamiltonians occur ubiquitously in many physical systems. Without any loss of generality, we can define a quadratic Hamiltonian as

$$\hat{H}(N) = \tfrac{1}{2}\alpha^T N \alpha \tag{3}$$

where N is a $2n \times 2n$ symmetric matrix and $\hat{H}$ is diagonalizable, namely we can find a unitary operator $\hat{U}$ so that $\hat{H} = \hat{U}(\sum_{i=1}^n c_i(a_i^\dagger a_i + a_i a_i^\dagger))\hat{U}^\dagger$ with $c_i > 0$. To facilitate further calculations, we also note that we can invoke the following formula [11]:

$$e^{t\hat{H}}\alpha e^{-t\hat{H}} = \alpha e^{tN\Sigma^{-1}}. \tag{4}$$

Writing

$$M(e^{t\hat{H}}) = e^{tN\Sigma^{-1}} \tag{5}$$

where $\Sigma = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and I is the $n \times n$ unity matrix, we denote the matrix representation of any operator $\hat{S}$ satisfying $\hat{S}\alpha\hat{S}^{-1} = \alpha M(\hat{S})$ as $M(\hat{S})$.

Consider the thermal states of the Hamiltonian $\hat{H}_i = \hat{H}(N_i) = \frac{1}{2}\alpha^T N_i \alpha$, $i = 1, 2$. The corresponding density operators are $\hat{\rho}_i = Z(\beta_i)e^{-\beta_i \hat{H}_i}$. The Bures fidelity is defined accordingly as

$$F = Z(\beta_1)Z(\beta_2)\left(\mathrm{tr}\sqrt{e^{-\frac{\beta_1}{2}\hat{H}_1}e^{-\beta_2\hat{H}_2}e^{-\frac{\beta_1}{2}\hat{H}_1}}\right)^2 \tag{6}$$

where $Z(\beta_i)$ is a normalization factor. Specifically, $Z(\beta_i) = \frac{1}{\mathrm{tr}\,\mathrm{e}^{-\beta_i \hat{H}_i}}$. In our notation, the matrix representation for the operator $\mathrm{e}^{-\frac{\beta_1}{2}\hat{H}_1}\mathrm{e}^{-\beta_2\hat{H}_2}\mathrm{e}^{-\frac{\beta_1}{2}\hat{H}_1}$ is

$$M(\mathrm{e}^{-\frac{\beta_1}{2}\hat{H}_1}\mathrm{e}^{-\beta_2\hat{H}_2}\mathrm{e}^{-\frac{\beta_1}{2}\hat{H}_1}) = \mathrm{e}^{\frac{\beta_1}{2}N_1\Sigma}\mathrm{e}^{\beta_2 N_2\Sigma}\mathrm{e}^{\frac{\beta_1}{2}N_1\Sigma}. \tag{7}$$

In general, it is not easy to calculate the fidelity, F, directly as defined above. However, it is possible to diagonalize the operator $\hat{H}_1$ with a unitary operator $\hat{U}$ so that

$$\hat{H}_1 = \tfrac{1}{2}\hat{U}_1 \sum_{i=1}^{n} \lambda_i (a_i^\dagger a_i + a_i a_i^\dagger)\hat{U}_1^\dagger. \tag{8}$$

If we denote

$$\hat{\Lambda}_1 = -\frac{\beta_1}{2}\sum_{i=1}^{n}\lambda_i(a_i^\dagger a_i + a_i a_i^\dagger) \tag{9}$$

we have

$$M(\hat{U}_1^\dagger \mathrm{e}^{-\beta_1\hat{H}_1}\hat{U}_1) = M(\mathrm{e}^{\hat{\Lambda}_1}) = \mathrm{e}^{-\beta_1 K_1} \tag{10}$$

where

$$K_1 = \begin{pmatrix} \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) & 0 \\ 0 & \mathrm{diag}(-\lambda_1, -\lambda_2, \ldots, -\lambda_n) \end{pmatrix}.$$

Moreover, we can denote $\mathrm{e}^{\frac{1}{2}\hat{\Lambda}_1}\hat{U}_1^\dagger \mathrm{e}^{-\beta_2\hat{H}_2}\hat{U}_1\mathrm{e}^{\frac{1}{2}\hat{\Lambda}_1}$ as $\hat{\Omega}$, so that the fidelity, F, becomes

$$F = Z(\beta_1)Z(\beta_2)\left(\mathrm{tr}\sqrt{\hat{U}_1\hat{\Omega}\hat{U}_1^\dagger}\right)^2 = Z(\beta_1)Z(\beta_2)\left(\mathrm{tr}\sqrt{\hat{\Omega}}\right)^2. \tag{11}$$

Since the matrix representation for $\hat{U}_1$ is $M(\hat{U}_1)$, the matrix representation for $\hat{\Omega}$ is

$$M(\mathrm{e}^{\frac{1}{2}\hat{\Lambda}_1})M^{-1}(\hat{U}_1)M(\mathrm{e}^{-\beta_2\hat{H}_2})M(\hat{U}_1^\dagger)M(\mathrm{e}^{\frac{1}{2}\hat{\Lambda}_1}). \tag{12}$$

Moreover, considering equations (4) and (5), we see that the operator $\hat{\Omega}'$ has the same matrix representation as the operator Ω with

$$\hat{\Omega}' = \exp[\tfrac{1}{2}\alpha^T \ln(M(\mathrm{e}^{\frac{1}{2}\hat{\Lambda}_1})M(\hat{U}_1^\dagger)M(\mathrm{e}^{-\beta_2\hat{H}_2})M(\hat{U}_1)M(\mathrm{e}^{\frac{1}{2}\hat{\Lambda}_1}))\Sigma\alpha]. \tag{13}$$

Applying Schur's lemma, the operators $\hat{\Omega}$ and $\hat{\Omega}'$ therefore differ at most by a constant factor. In the appendix, we have shown that this constant factor is unity. Thus $\hat{\Omega}' = \hat{\Omega}$.

We perform the rest of the calculations using $\hat{\Omega}'$ for the fidelity, F, by noting that

$$\sqrt{\hat{\Omega}'} = \exp[\tfrac{1}{4}\alpha^T \ln(M(\mathrm{e}^{\frac{1}{2}\hat{\Lambda}_1})M(\hat{U}_1^\dagger)M(\mathrm{e}^{-\beta_2\hat{H}_2})M(\hat{U}_1)M(\mathrm{e}^{\frac{1}{2}\hat{\Lambda}_1}))\Sigma\alpha] \tag{14}$$

$$\equiv \exp\left[\tfrac{1}{2}\alpha^T \ln\sqrt{M(\mathrm{e}^{\frac{1}{2}\hat{\Lambda}_1})M(\hat{U}_1^\dagger)M(\mathrm{e}^{-\beta_2\hat{H}_2})M(\hat{U}_1)M(\mathrm{e}^{\frac{1}{2}\hat{\Lambda}_1})}\Sigma\alpha\right]. \tag{15}$$

To take the trace of the operator, $\sqrt{\hat{\Omega}'}$, we can use the result in [12] which states that $\mathrm{tr}\,\mathrm{e}^{-\beta_i\hat{H}_i} = (\sqrt{|\det(\mathrm{e}^{-\beta_i N_i\Sigma^{-1}} - I)|})^{-1}$, giving

$$\mathrm{tr}\sqrt{\hat{\Omega}'} = \frac{1}{\sqrt{\left|\det\left(\sqrt{M(\mathrm{e}^{\frac{\hat{\Lambda}_1}{2}})M(\hat{U}_1^\dagger)M(\mathrm{e}^{-\beta_2\hat{H}_2})M(\hat{U}_1)M(\mathrm{e}^{\frac{\hat{\Lambda}_1}{2}})} - I\right)\right|}}. \tag{16}$$

Since $M(\hat{U}_1)M(\mathrm{e}^{\frac{1}{2}\hat{\Lambda}_1})M(\hat{U}_1^\dagger) = M(\mathrm{e}^{-\beta_1\hat{H}_1}) = \mathrm{e}^{-\beta_1 N_1\Sigma^{-1}}$, we can express equation (16) succinctly as

$$\mathrm{tr}\sqrt{\hat{\Omega}'} = \frac{1}{\sqrt{\left|\det\left(\sqrt{\mathrm{e}^{-\frac{\beta_1}{2}N_1\Sigma^{-1}}\mathrm{e}^{-\beta_2 N_2\Sigma^{-1}}\mathrm{e}^{-\frac{\beta_1}{2}N_1\Sigma^{-1}}} - I\right)\right|}}. \tag{17}$$

As a final step, we can then use equation (11) to obtain the formula for Bures fidelity as

$$F = \left| \frac{\det(e^{-\beta_1 N_1 \Sigma^{-1}} - I)\det(e^{-\beta_2 N_2 \Sigma^{-1}} - I)}{\det\left(\sqrt{e^{-\frac{\beta_1}{2} N_1 \Sigma^{-1}} e^{-\beta_2 N_2 \Sigma^{-1}} e^{-\frac{\beta_1}{2} N_1 \Sigma^{-1}}} - I\right)} \right|^{\frac{1}{2}}. \tag{18}$$

Equation (18) therefore allows us to compute the Bures fidelity directly from the Hamiltonians of any two thermal systems.

3. Applications

3.1. One-dimensional squeezed thermal states

It is illuminating to illustrate our approach with some instructive but non-trivial examples. Our first example is the fidelity of the one-dimensional squeezed thermal states, investigated earlier by Twamley [2]. Although the result for this example is well known now, the example nevertheless serves as a useful check for our approach. For the one-dimensional squeezed thermal states, one considers the density operator

$$\rho_i = Z(\beta_i)\hat{S}_i\hat{T}_i\hat{S}_i^\dagger \equiv e^{-\beta_i \hat{H}_i} \tag{19}$$

$i = 1, 2$, $S_i = \exp[\frac{1}{2}(\zeta_i^* a^{\dagger 2} - \zeta_i a^2)]$ and $T_i = \exp[-\frac{\beta_i}{2}(a^\dagger a + a a^\dagger)]$. Using our notation described in the previous section, it is not difficult to see that

$$M(\hat{S}_i) = \begin{pmatrix} \cosh r_i & -e^{i\theta}\sinh r_i \\ -e^{-i\theta}\sinh r_i & \cosh r_i \end{pmatrix} \qquad M(\hat{S}^\dagger) = \begin{pmatrix} \cosh r_i & e^{i\theta}\sinh r_i \\ e^{-i\theta}\sinh r_i & \cosh r_i \end{pmatrix}$$

and $r_i = |\zeta_i|, \theta_i = \zeta_i / r_i$. Moreover, in this representation, the matrix $M(\hat{T}_i) = \begin{pmatrix} e^{\beta_i} & 0 \\ 0 & e^{-\beta_i} \end{pmatrix}$ is diagonal. Using the Baker–Campbell–Hausdorff (BCH) relation, we can obtain the following formula for $\hat{\rho}_i$:

$$\hat{\rho}_i = \exp\left[\tfrac{1}{2}\alpha^T M(\hat{S}_i)\begin{pmatrix} 0 & -\beta_i \\ -\beta_i & 0 \end{pmatrix} M(\hat{S}_i)^T \alpha\right]. \tag{20}$$

From a quick comparison with the form of N_i in equation (3) and using the result in equation (18), we see that

$$N_i = M(\hat{S}_i)\begin{pmatrix} 0 & -\beta_i \\ -\beta_i & 0 \end{pmatrix} M(\hat{S}_i)^T.$$

Thus

$$e^{-\beta_i N_i \Sigma^{-1}} = M(\hat{\rho}_i) = \begin{pmatrix} e_i^{\beta}\cosh^2 r_i - e^{-\beta_i}\sinh^2 r_i & e^{i\theta_i}\sinh 2r_i \sinh\beta_i \\ -e^{-i\theta_i}\sinh 2r_i \sinh\beta_i & e^{-\beta_i}\cosh^2 r_i - e^{\beta_i}\sinh^2 r_i \end{pmatrix}. \tag{21}$$

A short calculation then gives

$$Z(\beta_i) = \det(e^{-\beta_i N_i \Sigma^{-1}} - I) = (e^{\beta_i} - 1)(e^{-\beta_i} - 1) \tag{22}$$

and

$$\det\left(\sqrt{e^{-\frac{\beta_1}{2} N_1 \Sigma^{-1}} e^{-\beta_2 N_2 \Sigma^{-1}} e^{-\frac{\beta_1}{2} N_1 \Sigma^{-1}}} - I\right) = (e^{\beta_3/2} - 1)(e^{-\beta_3/2} - 1) \tag{23}$$

where β_3 must satisfy

$$\begin{aligned} 2\cosh\beta_3 &= \mathrm{tr}(e^{-\frac{\beta_1}{2} N_1 \Sigma^{-1}} e^{-\beta_2 N_2 \Sigma^{-1}} e^{-\frac{\beta_1}{2} N_1 \Sigma^{-1}}) = \mathrm{tr}(M(\hat{\rho}_1)M(\hat{\rho}_2)) \\ &= (e^{\beta_1}\cosh^2 r_1 - e^{-\beta_1}\sinh^2 r_1)(e^{\beta_2}\cosh^2 r_2 - e^{-\beta_2}\sinh^2 r_2) \\ &\quad -2\cosh(i\Delta\theta)\sinh\beta_1 \sinh\beta_2 \sinh 2r_1 \sinh 2r_2 \\ &\quad +(e^{-\beta_1}\cosh^2 r_1 - e^{\beta_1}\sinh^2 r_1)(e^{-\beta_2}\cosh^2 r_2 - e^{\beta_2}\sinh^2 r_2) \end{aligned} \tag{24}$$

which can be simplified further as

$$\cosh\beta_3 = \cosh(\beta_1+\beta_2)(\cosh^2 r_1\cosh^2 r_2 + \sinh^2 r_1 \sinh^2 r_2) \\ - \cosh(\beta_2-\beta_1)(\sinh^2 r_1\cosh^2 r_2 + \cosh^2 r_1 \sinh^2 r_2) \\ - \cosh \mathrm{i}\Delta\theta \sinh\beta_1 \sinh\beta_2 \sinh 2r_1 \sinh 2r_2 \tag{25}$$

$$= \cosh(\beta_1+\beta_2)\left[\cosh^2(r_1+r_2)\sin^2\frac{\Delta\theta}{2} + \cosh^2(r_2-r_1)\cos^2\frac{\Delta\theta}{2}\right] \\ - \cosh(\beta_2-\beta_1)\left[\sinh^2(r_1+r_2)\sin^2\frac{\Delta\theta}{2} + \sinh^2(r_2-r_1)\cos^2\frac{\Delta\theta}{2}\right]. \tag{26}$$

Thus, we have

$$F = -\frac{(\mathrm{e}^{\beta_1}-1)(\mathrm{e}^{-\beta_1}-1)(\mathrm{e}^{\beta_2}-1)(\mathrm{e}^{-\beta_2}-1)}{(\mathrm{e}^{\beta_3/2}-1)(\mathrm{e}^{-\beta_3/2}-1)} \tag{27}$$

or equivalently

$$F = \frac{2\sinh\frac{\beta_1}{2}\sinh\frac{\beta_2}{2}}{\cosh\frac{\beta_3}{2}-1}. \tag{28}$$

The definition of θ in Twamley's paper [5] differs from the definition of φ in [13]. Taking this into consideration, we see that our results essentially reproduce Twamley's expression [5].

3.2. Two-mode squeezed thermal states

Having considered the one-mode squeezed thermal states, it is natural to extend the previous application to the two-mode case. As we have mentioned earlier, in experimental settings, it is often more convenient to obtain two-mode squeezed thermal states using a non-degenerate parametric amplifier. For the two-mode squeezed thermal states, the density operator is given by

$$\rho_i = Z(\beta_i)\hat{S}_i\hat{T}_i\hat{S}_i^\dagger \equiv \mathrm{e}^{-\beta_i\hat{H}_i} \tag{29}$$

where $i = 1, 2$, $S_i = \exp[(\zeta_i^* a_1^\dagger a_2^\dagger - \zeta_i a_1 a_2)]$ and $T_i = \exp[-\frac{\beta_i}{2}\sum_{j=1}^2 (a_j^\dagger a_j + a_j a_j^\dagger)]$. In this case,

$$M(\hat{S}_i) = \begin{pmatrix} \cosh r_i I & -\mathrm{e}^{\mathrm{i}\theta}\sinh r_i\sigma \\ -\mathrm{e}^{-\mathrm{i}\theta}\sinh r_i\sigma & \cosh r_i I \end{pmatrix}$$

and

$$M(\hat{S}^\dagger) = \begin{pmatrix} \cosh r_i I & \mathrm{e}^{\mathrm{i}\theta}\sinh r_i\sigma \\ \mathrm{e}^{-\mathrm{i}\theta}\sinh r_i\sigma & \cosh r_i I \end{pmatrix}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with $r_i = |\zeta_i|$, $\theta_i = \zeta_i/r_i$. Moreover, we have $M(\hat{T}_i) = \begin{pmatrix} \mathrm{e}^{\beta_i} I & 0 \\ 0 & \mathrm{e}^{-\beta_i} I \end{pmatrix}$. Using the BCH relation, we obtain the following formula for ρ_i:

$$\rho_i = \exp\left[\tfrac{1}{2}\alpha^T M(\hat{S}_i)\begin{pmatrix} 0 & -\beta_i I \\ -\beta_i I & 0 \end{pmatrix} M(\hat{S}_i)^T\alpha\right]. \tag{30}$$

By comparing with the definition of N_i in equation (18), we see that in this case

$$N_i = M(\hat{S}_i)\begin{pmatrix} 0 & -\beta_i I \\ -\beta_i I & 0 \end{pmatrix} M(\hat{S}_i)^T.$$

Finally, using equation (18), we obtain the Bures fidelity for two-mode squeezed thermal states as

$$F = \left(\frac{2 \sinh \frac{\beta_1}{2} \sinh \frac{\beta_2}{2}}{\cosh \frac{\beta_3}{2} - 1} \right)^2 \tag{31}$$

where β_3 is the same as the definition given in equation (26). It is instructive to compare this formula with the one-mode squeezed thermal state in equation (28). Despite the seeming triviality in the results, this example highlights a physical property of Bures fidelity for factorizable systems. For truly non-trivial, non-factorizable examples, we need to consider the jump oscillators and the Liu–Tombesi oscillators.

3.3. Jump oscillators

As a non-trivial example, we consider a time dependent oscillator in which the frequency encounters a jump at some finite time. Jump oscillators have been extensively investigated in the past few years [14]. For pure states, the calculation of the projection between states at different times allows us to determine the properties associated with time evolution. This feature has been demonstrated, e.g. in [15], where Frank–Condon factors are computed. For mixed states, the same problem is handled using density operators and one computes the Bures fidelity between the density matrices at different times for time evolution. The jump Hamiltonian for the two-dimensional harmonic oscillator system is

$$\hat{H}_1 = \tfrac{1}{2} p^T p + \tfrac{1}{2} x^T x \tag{32}$$

at $t = 0$ and

$$\hat{H}_2 = \tfrac{1}{2} p^T p + \tfrac{1}{2} x^T x + \lambda x_1 x_2 \tag{33}$$

at $t > 0$ where $x^T = (x_1, x_2)$, $p^T = (p_1, p_2)$. We can now calculate the Bures fidelity between thermal states at $t = 0$ and at arbitrary t for the temperature $\frac{k_B}{\beta}$. We know that the density operator takes the form $\hat{\rho}_1 = \hat{\rho}(t = 0) = Z(\beta) \mathrm{e}^{-\beta H_1}$, $Z(\beta) = (2 \sinh \frac{\beta}{2})^2$ and $\rho_2 = \rho(t > 0) = \mathrm{e}^{-\mathrm{i}\hat{H}_1 t} \hat{\rho}_1 \mathrm{e}^{\mathrm{i}\hat{H}_1 t}$. We convert the operators into Fock space by using the following relation:

$$(q^T, p^T) = (a^\dagger, a^T) \frac{1}{\sqrt{2}} \begin{pmatrix} I & \mathrm{i}I \\ I & -\mathrm{i}I \end{pmatrix} = (a^\dagger, a^T) K \tag{34}$$

where I is a two-dimensional unit matrix. With this conversion, we can use equation (18) to obtain the result for

$$\left(\mathrm{tr} \sqrt{\mathrm{e}^{-\frac{1}{2}\beta \hat{H}_1} \mathrm{e}^{-\mathrm{i}\hat{H}_2 t} \mathrm{e}^{-\beta \hat{H}_1} \mathrm{e}^{\mathrm{i}\hat{H}_2 t} \mathrm{e}^{-\frac{1}{2}\beta \hat{H}_1}} \right)^2$$

by defining the following matrices:

$$N_1 = K \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} K^T \tag{35}$$

and

$$N_2 = K \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \lambda \\ 0 & 0 & \lambda & 1 \end{pmatrix} K^T. \tag{36}$$

For convenience, we set $\beta_1 = \beta_2 = \beta$. The rest is a straightforward calculation using equation (18). After some tedious but straightforward calculations, we finally obtain the fidelity as

$$F = \frac{4\sinh^4\frac{\beta}{2}}{(\cosh\frac{\gamma}{2} - 1)^2} \tag{37}$$

where γ is defined as

$$\cosh\gamma = \frac{1}{16(1-\lambda^2)}\bigg[-2\lambda^2 + \lambda^2(1+\lambda)\cos\left(2t\sqrt{1-\lambda}\right) + \lambda^2(1-\lambda)\cos\left(2t\sqrt{1+\lambda}\right)$$
$$+\cosh 2\beta\bigg(16 - 14\lambda^2 - \lambda^2(1+\lambda)\cos\left(2t\sqrt{1-\lambda}\right)$$
$$-\lambda^2(1-\lambda)\cos\left(2t\sqrt{1+\lambda}\right)\bigg)\bigg]. \tag{38}$$

Despite the form of equation (37), the Bures fidelity in this example is non-trivial since physically it is not possible to diagonalize both the Hamiltonians $\hat{H}_1$ and $\hat{H}_2$ simultaneously. In other words, it is not easy to factorize the states $\exp(-\beta H_1)$ and $\exp(-\beta H_2)$ at the same time. Moreover, the one-mode situation does not arise since physically it makes no sense to set $\lambda = 0$, yielding $F = 1$.

3.4. Liu–Tombesi Hamiltonian

As a final application of our approach, we consider the thermal states of the following Hamiltonian [16]:

$$\hat{A}_i = \mathrm{i}\zeta_i a_1^\dagger a_2^\dagger - \mathrm{i}\zeta_i^* a_1 a_2 + \frac{y_i}{2}(a_1^\dagger a_1 + a_1 a_1^\dagger + a_2^\dagger a_2 + a_2 a_2^\dagger) \tag{39}$$

where y_i is a real number and $y_i > |\zeta_i|, i = 1, 2$. This Hamiltonian is related to the generalized two-dimensional squeezed states. The corresponding density operator is

$$\hat{\rho}_i = Z_i \mathrm{e}^{-\beta\hat{A}_i} \tag{40}$$

($i = 1, 2$). To calculate the Bures fidelity between the density operators, ρ_1 and ρ_2, we need to rewrite A_i as

$$\hat{A}_i = \tfrac{1}{2}\alpha^T \begin{pmatrix} 0 & \mathrm{i}\zeta_i & y_i & 0 \\ \mathrm{i}\zeta_i & 0 & 0 & y_i \\ y_i & 0 & 0 & -\mathrm{i}\zeta_i^* \\ 0 & y_i & -\mathrm{i}\zeta_i^* & 0 \end{pmatrix} \alpha \tag{41}$$

where $\alpha^T = (a_1^\dagger, a_2^\dagger, a_1, a_2)$, giving

$$N_i = \begin{pmatrix} 0 & \mathrm{i}\zeta_i & y_i & 0 \\ \mathrm{i}\zeta_i & 0 & 0 & y_i \\ y_i & 0 & 0 & -\mathrm{i}\zeta_i^* \\ 0 & y_i & -\mathrm{i}\zeta_i^* & 0 \end{pmatrix}.$$

Computing directly using equation (18), we obtain an expression for Bures fidelity for the density operators, $\hat{\rho}_1$ and $\hat{\rho}_2$, as

$$F = \frac{(\mathrm{e}^{\beta_1\sqrt{y_1^2-|\zeta_1|^2}} - 1)(\mathrm{e}^{-\beta_1\sqrt{y_1^2-|\zeta_1|^2}} - 1)(\mathrm{e}^{\beta_2\sqrt{y_2^2-|\zeta_2|^2}} - 1)(\mathrm{e}^{-\beta_2\sqrt{y_2^2-|\zeta_2|^2}} - 1)}{4(\cosh\frac{\beta'}{2} - 1)^2} \tag{42}$$

which is equivalent to

$$F = \left(\frac{2\sinh\frac{\beta_1\chi_1}{2}\sinh\frac{\beta_2\chi_2}{2}}{\cosh\frac{\beta'}{2} - 1} \right)^2. \tag{43}$$

After defining $\chi_1 = \sqrt{y_1^2 - |\zeta_1|^2}$ and $\chi_2 = \sqrt{y_2^2 - |\zeta_2|^2}$, we have the following formula for β':

$$\cosh\beta' = \cosh\beta_1\chi_1\cosh\beta_2\chi_2 + \frac{y_1y_2 - \mathrm{Re}\,(\zeta_1^*\zeta_2)}{\chi_1\chi_2}\sinh(\beta_1\chi_1)\sinh\beta_2\chi_2. \tag{44}$$

It is interesting to note that equation (43) is again a non-trivial example since the cross-terms in the Hamiltonian in equation (39) involving the parameters ζ_i $(i = 1, 2)$ cannot be set to zero. Thus, it is not possible to diagonalize the two states $\exp(-\beta H_1)$ and $\exp(-\beta H_2)$ using the same unitary matrices.

4. Discussion

In this paper, we have the Bures fidelities for multi-mode oscillators. A subtle point to note from our computation is that, even if the final explicit expression for the Bures fidelities for two-mode oscillators appear to be factorizable, the states may not be uncorrelated disentangled or factorizable states. In other words, for the computation of the Bures fidelity of any two states of a multi-mode oscillator, unless we can diagonalize the two states at the same time, it is not clear if we have a factorizable situation. Moreover, while it is true that factorizable states leads to factorizable fidelities, the converse is not true.

Finally, let us reiterate the main points. Bures fidelities for single-mode squeezed thermal states have been extensively explored in the literature. However, experimentally, multi-mode squeezed states are sometimes more easily reproduced, for instance in a non-degenerate parametric homodyne amplifier. It is therefore natural to extend the investigation of the single-mode case to the multi-mode case. Existing techniques using group-theoretic methods and differential equations may not be easily adaptable for the multi-mode extension. Nevertheless, we have shown in this paper that it is possible to compute the Bures fidelity for the multi-mode case using the technique of exponential quadratic operators [7, 18].

Indeed, once the Hamiltonian for the system has been cast into the form in equation (3), the Bures fidelity can be computed using the formula in equation (18). More specifically, one first writes the density matrices $\hat{\rho}_i$ as $Z(\beta_i)\mathrm{e}^{-\frac{\beta_i}{2}\alpha_i^T N_i \alpha}$, identifies the matrices N_i and computes the Bures fidelity using equation (18).

To illustrate the usefulness of our technique, we have applied the method to some instructive examples and derived the explicit expression for the Bures fidelity in each case. In particular, we have considered at least two non-trivial, non-factorizable cases: a two-dimensional harmonic oscillator subjected to a jump Hamiltonian and a two-dimensional squeezed oscillator. Finally, we note that it would be interesting to explore the Bures metrics and the prior probability distributions and investigate the Fisher information metrics [17] for these multi-mode systems.

Appendix

In this appendix, we show that the constant factor for the operators Ω and Ω' is unity. To determine this factor, we only need to compare the value of $\langle 0|\hat{\Omega}|0\rangle$ and $\langle 0|\hat{\Omega}'|0\rangle$. Suppose

$$M(\hat{\Omega}') = \begin{pmatrix} A' & B' \\ D' & C' \end{pmatrix}. \tag{45}$$

Using the method used in the appendix of [18],

$$\langle 0|\hat{\Omega}'|0\rangle = \det(C')^{-\frac{1}{2}}. \tag{46}$$

Furthermore, we have

$$\langle 0|\Omega|0\rangle = \exp\left[\sum_{i=0}^{n} -\tfrac{1}{2}\beta_1\lambda_i\right]\langle 0|\hat{U}_1^\dagger \mathrm{e}^{-\beta_2\hat{H}_2}\hat{U}_1|0\rangle. \tag{47}$$

We next consider $\hat{U}_1^\dagger \alpha^T \hat{U}_1 = \alpha^T M^{-1}(\hat{U}_1)$ and $\hat{U}_1^\dagger \alpha \hat{U}_1 = {M^{-1}}^T(\hat{U}_1)\alpha$. Thus

$$\hat{U}_1^\dagger \mathrm{e}^{-\beta_2\hat{H}_2}\hat{U}_1 = \mathrm{e}^{-\beta_2\alpha^T M^{-1}(\hat{U}_1)N_2 {M^{-1}}^T(\hat{U}_1)\alpha}. \tag{48}$$

To simplify calculations, we can decompose the $2n \times 2n$ matrices into $n \times n$ block matrices by defining the matrix representations of the operator on the rhs of equation (48) as

$$\mathrm{e}^{-\beta_2 M^{-1}(\hat{U}_1)N_2 {M^{-1}}^T(\hat{U}_1)\Sigma^{-1}} = \begin{pmatrix} A & B \\ D & C \end{pmatrix} \tag{49}$$

so that we obtain

$$\langle 0|\Omega|0\rangle = \exp\left[-\frac{\beta_1}{2}\sum_{i=0}^{n}\lambda_i\right]\det C^{-\frac{1}{2}}. \tag{50}$$

Since the matrix M is symplectic, we have

$$M^T = \Sigma M^{-1}\Sigma^{-1}. \tag{51}$$

Thus

$$\mathrm{e}^{-\beta_2 M^{-1}(\hat{U}_1)N_2 {M^{-1}}^T(\hat{U}_1)\Sigma^{-1}} = M^{-1}(\hat{U}_1)\mathrm{e}^{-\beta_2 N_2\Sigma^{-1}}M(\hat{U}_1). \tag{52}$$

Comparing equations (12) and (52), we easily see the relation between matrices $\begin{pmatrix} A & B \\ D & C \end{pmatrix}$ and $\begin{pmatrix} A' & B' \\ D' & C' \end{pmatrix}$ where A, B, C, D and their primes are $n \times n$ matrices. Writing

$$\begin{pmatrix} A' & B' \\ D' & C' \end{pmatrix} = M(\mathrm{e}^{\frac{1}{2}\hat{\Lambda}_1})\begin{pmatrix} A & B \\ D & C \end{pmatrix}M(\mathrm{e}^{\frac{1}{2}\hat{\Lambda}_1}) \tag{53}$$

we obtain

$$C' = \begin{pmatrix} \mathrm{e}^{\frac{\lambda_1}{2}\beta_1} & 0 & 0 & 0 \\ 0 & \mathrm{e}^{\frac{\lambda_2}{2}\beta_1} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \mathrm{e}^{\frac{\lambda_n}{2}\beta_1} \end{pmatrix} C \begin{pmatrix} \mathrm{e}^{\frac{\lambda_1}{2}\beta_1} & 0 & 0 & 0 \\ 0 & \mathrm{e}^{\frac{\lambda_2}{2}\beta_1} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \mathrm{e}^{\frac{\lambda_n}{2}\beta_1} \end{pmatrix} \equiv \Gamma C' \Gamma.$$

Finally, using equation (50), we obtain the following important result:

$$\langle 0|\hat{\Omega}|0\rangle = \exp\left[-\frac{\beta_1}{2}\sum_{i=0}^{n}\lambda_i\right]\det(\Gamma^{-1}C'\Gamma^{-1})^{-\frac{1}{2}} = \det(C')^{-\frac{1}{2}} = \langle 0|\hat{\Omega}'|0\rangle. \tag{54}$$

References

[1] Jozsa R 1994 *J. Mod. Opt.* **41** 2315

[2] Barnum H, Caves C M and Fuchs C A 1996 *Phys. Rev. Lett.* **76** 2818
Buzek V and Hillary M 1996 *Phys. Rev.* A **54** 1844
Lo H K and Chau H F 1997 *Phys. Rev. Lett.* **78** 3410

[3] Deuar P and Munro W J 2000 *Preprint* quant-ph/0003054

[4] Hofmann H F, Ide T, Kobayashi T and Furusawa A 2000 *Preprint* quant-ph/0003053

[5] Twamley J 1996 *J. Phys. A: Math. Gen.* **29** 3723
[6] Scutaru H 1998 *J. Phys. A: Math. Gen.* **31** 3659
[7] Wang X B, Oh C H and Kwek L C 1998 *Phys. Rev.* A **58** 4186
[8] Paraoanu Gh S and Scutaru H 1998 *Phys. Rev.* A **58** 869
[9] Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (Cambridge: Cambridge University Press)
[10] Ou Z Y, Hong C K and Mandel L 1987 *J. Opt. Soc. Am.* B **4** 1574
Yurke B 1995 *Phys. Rev.* A **32** 300
Yurke B 1985 *Phys. Rev.* A **32** 311
[11] Balian R and Brezin E 1969 *Nuovo Cimento* B **64** 37
[12] Friesner R, Pettitt M and Jean J M 1985 *J. Chem. Phys.* **82** 2918
Pan J W, Dong Q X, Zhang Y D, Hou G and Wang X B 1997 *Phys. Rev.* E **56** 2553
[13] Schumaker B L 1986 *Phys. Rep.* **135** 317
[14] Janszky J and Yushin Y Y 1986 *Opt. Commun.* **59** 151
Graham R 1987 *J. Mod. Opt.* **34** 873
Fan H Y and Zaidi R 1988 *Phys. Rev.* A **37** 2985
Ma X and Rhodes W 1989 *Phys. Rev.* A **39** 1941
Lo C F 1990 *J. Phys. A: Math. Gen.* **23** 1155
[15] Ballhausen C J 1993 *Chem. Phys. Lett.* **201** 269
[16] Liu W S and Tombesi P 1993 *Quantum Opt.* **5** 181
[17] Braunstein S L and Caves C M 1994 *Phys. Rev. Lett.* **72** 3439
[18] Wang X B, Yu S X and Zhang Y D 1994 *J. Phys. A: Math. Gen.* **27** 6563